# ModulePresentationsForCAP

## Category R-pres for CAP

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# Chapter 1

# Module Presentations

## 1.1 Functors

### 1.1.1 FunctorStandardModuleLeft (for IsHomalgRing)

▷ `FunctorStandardModuleLeft(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is a functor which takes a left presentation as input and computes its standard presentation.

### 1.1.2 FunctorStandardModuleRight (for IsHomalgRing)

▷ `FunctorStandardModuleRight(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is a functor which takes a right presentation as input and computes its standard presentation.

### 1.1.3 FunctorGetRidOfZeroGeneratorsLeft (for IsHomalgRing)

▷ `FunctorGetRidOfZeroGeneratorsLeft(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is a functor which takes a left presentation as input and gets rid of the zero generators.

### 1.1.4 FunctorGetRidOfZeroGeneratorsRight (for IsHomalgRing)

▷ `FunctorGetRidOfZeroGeneratorsRight(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is a functor which takes a right presentation as input and gets rid of the zero generators.

### 1.1.5 FunctorLessGeneratorsLeft (for IsHomalgRing)

▷ `FunctorLessGeneratorsLeft(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is functor which takes a left presentation as input and computes a presentation having less generators.

### 1.1.6 FunctorLessGeneratorsRight (for IsHomalgRing)

▷ `FunctorLessGeneratorsRight(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$. The output is functor which takes a right presentation as input and computes a presentation having less generators.

### 1.1.7 FunctorDualLeft (for IsHomalgRing)

▷ `FunctorDualLeft(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$ that has an involution function. The output is functor which takes a left presentation *M* as input and computes its Hom(M, R) as a left presentation.

### 1.1.8 FunctorDualRight (for IsHomalgRing)

▷ `FunctorDualRight(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$ that has an involution function. The output is functor which takes a right presentation *M* as input and computes its Hom(M, R) as a right presentation.

### 1.1.9 FunctorDoubleDualLeft (for IsHomalgRing)

▷ `FunctorDoubleDualLeft(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$ that has an involution function. The output is functor which takes a left presentation *M* as input and computes its *Hom( Hom(M, R), R )* as a left presentation.

### 1.1.10 FunctorDoubleDualRight (for IsHomalgRing)

▷ `FunctorDoubleDualRight(R)` (attribute)

**Returns:** a functor

The argument is a homalg ring $R$ that has an involution function. The output is functor which takes a right presentation *M* as input and computes its *Hom( Hom(M, R), R )* as a right presentation.

## 1.2 GAP Categories

### 1.2.1 IsLeftOrRightPresentationMorphism (for IsCapCategoryMorphism)

▷ `IsLeftOrRightPresentationMorphism(object)` (filter)

**Returns:** `true` or `false`

The GAP category of morphisms in the category of left or right presentations.

### 1.2.2 IsLeftPresentationMorphism (for IsLeftOrRightPresentationMorphism)

▷ `IsLeftPresentationMorphism(object)` (filter)

**Returns:** `true` or `false`

The GAP category of morphisms in the category of left presentations.

### 1.2.3 IsRightPresentationMorphism (for IsLeftOrRightPresentationMorphism)

▷ `IsRightPresentationMorphism(object)` (filter)

**Returns:** `true` or `false`

The GAP category of morphisms in the category of right presentations.

### 1.2.4 IsLeftOrRightPresentation (for IsCapCategoryObject)

▷ `IsLeftOrRightPresentation(object)` (filter)

**Returns:** `true` or `false`

The GAP category of objects in the category of left presentations or right presentations.

### 1.2.5 IsLeftPresentation (for IsLeftOrRightPresentation)

▷ `IsLeftPresentation(object)` (filter)

**Returns:** `true` or `false`

The GAP category of objects in the category of left presentations.

### 1.2.6 IsRightPresentation (for IsLeftOrRightPresentation)

▷ `IsRightPresentation(object)` (filter)

**Returns:** `true` or `false`

The GAP category of objects in the category of right presentations.

## 1.3 Constructors

### 1.3.1 PresentationMorphism (for IsLeftOrRightPresentation, IsHomalgMatrix, IsLeftOrRightPresentation)

▷ `PresentationMorphism(A, M, B)` (operation)

**Returns:** a morphism in $\mathrm{Hom}(A,B)$

The arguments are an object $A$, a homalg matrix $M$, and another object $B$. $A$ and $B$ shall either both be objects in the category of left presentations or both be objects in the category of right presentations. The output is a morphism $A \to B$ in the the category of left or right presentations whose underlying matrix is given by $M$.

### 1.3.2 AsMorphismBetweenFreeLeftPresentations (for IsHomalgMatrix)

▷ `AsMorphismBetweenFreeLeftPresentations(m)` (attribute)

**Returns:** a morphism in $\mathrm{Hom}(F^r,F^c)$

The argument is a homalg matrix $m$. The output is a morphism $F^r \to F^c$ in the the category of left presentations whose underlying matrix is given by $m$, where $F^r$ and $F^c$ are free left presentations of ranks given by the number of rows and columns of $m$.

### 1.3.3 AsMorphismBetweenFreeRightPresentations (for IsHomalgMatrix)

▷ `AsMorphismBetweenFreeRightPresentations(m)` (attribute)

**Returns:** a morphism in $\mathrm{Hom}(F^c, F^r)$

The argument is a homalg matrix $m$. The output is a morphism $F^c \to F^r$ in the the category of right presentations whose underlying matrix is given by $m$, where $F^r$ and $F^c$ are free right presentations of ranks given by the number of rows and columns of $m$.

### 1.3.4 AsLeftPresentation (for IsHomalgMatrix)

▷ `AsLeftPresentation(M)` (operation)

**Returns:** an object

The argument is a homalg matrix $M$ over a ring $R$. The output is an object in the category of left presentations over $R$. This object has $M$ as its underlying matrix.

### 1.3.5 AsRightPresentation (for IsHomalgMatrix)

▷ `AsRightPresentation(M)` (operation)

**Returns:** an object

The argument is a homalg matrix $M$ over a ring $R$. The output is an object in the category of right presentations over $R$. This object has $M$ as its underlying matrix.

### 1.3.6 AsLeftOrRightPresentation

▷ `AsLeftOrRightPresentation(M, l)` (function)

**Returns:** an object

The arguments are a homalg matrix $M$ and a boolean $l$. If $l$ is `true`, the output is an object in the category of left presentations. If $l$ is `false`, the output is an object in the category of right presentations. In both cases, the underlying matrix of the result is $M$.

### 1.3.7 FreeLeftPresentation (for IsInt, IsHomalgRing)

▷ `FreeLeftPresentation(r, R)` (operation)

**Returns:** an object

The arguments are a non-negative integer $r$ and a homalg ring $R$. The output is an object in the category of left presentations over $R$. It is represented by the $0 \times r$ matrix and thus it is free of rank $r$.

### 1.3.8 FreeRightPresentation (for IsInt, IsHomalgRing)

▷ `FreeRightPresentation(r, R)` (operation)

**Returns:** an object

The arguments are a non-negative integer $r$ and a homalg ring $R$. The output is an object in the category of right presentations over $R$. It is represented by the $r \times 0$ matrix and thus it is free of rank $r$.

### 1.3.9 UnderlyingMatrix (for IsLeftOrRightPresentation)

▷ `UnderlyingMatrix(A)` (attribute)

**Returns:** a homalg matrix

The argument is an object $A$ in the category of left or right presentations over a homalg ring $R$. The output is the underlying matrix which presents $A$.

### 1.3.10 UnderlyingHomalgRing (for IsLeftOrRightPresentation)

▷ `UnderlyingHomalgRing(A)` (attribute)

**Returns:** a homalg ring

The argument is an object $A$ in the category of left or right presentations over a homalg ring $R$. The output is $R$.

### 1.3.11 Annihilator (for IsLeftOrRightPresentation)

▷ `Annihilator(A)` (attribute)

**Returns:** a morphism in $\mathrm{Hom}(I,F)$

The argument is an object $A$ in the category of left or right presentations. The output is the embedding of the annihilator $I$ of $A$ into the free module $F$ of rank 1. In particular, the annihilator itself is seen as a left or right presentation.

### 1.3.12 LeftPresentationsAsFreydCategoryOfCategoryOfRows (for IsHomalgRing)

▷ `LeftPresentationsAsFreydCategoryOfCategoryOfRows(R)` (operation)

**Returns:** a category

The argument is a homalg ring $R$. The output is the category of left presentations over $R$, constructed internally as the `FreydCategory` of the `CategoryOfRows` of `R`. Only available if the package `FreydCategoriesForCAP` is available.

### 1.3.13 RightPresentationsAsFreydCategoryOfCategoryOfColumns (for IsHomalgRing)

▷ `RightPresentationsAsFreydCategoryOfCategoryOfColumns(R)` (operation)

**Returns:** a category

The argument is a homalg ring $R$. The output is the category of right presentations over $R$, constructed internally as the `FreydCategory` of the `CategoryOfColumns` of `R`. Only available if the package `FreydCategoriesForCAP` is available.

### 1.3.14 LeftPresentations (for IsHomalgRing)

▷ `LeftPresentations(R)` (attribute)

**Returns:** a category

The argument is a homalg ring $R$. The output is the category of left presentations over $R$.

### 1.3.15 RightPresentations (for IsHomalgRing)

▷ `RightPresentations(R)` (attribute)

**Returns:** a category

The argument is a homalg ring $R$. The output is the category of right presentations over $R$.

## 1.4 Attributes

### 1.4.1 UnderlyingHomalgRing (for IsLeftOrRightPresentationMorphism)

▷ `UnderlyingHomalgRing(R)` (attribute)

**Returns:** a homalg ring

The argument is a morphism $\alpha$ in the category of left or right presentations over a homalg ring $R$. The output is $R$.

### 1.4.2 UnderlyingMatrix (for IsLeftOrRightPresentationMorphism)

▷ `UnderlyingMatrix(alpha)` (attribute)

**Returns:** a homalg matrix

The argument is a morphism $\alpha$ in the category of left or right presentations. The output is its underlying homalg matrix.

## 1.5 Non-Categorical Operations

### 1.5.1 StandardGeneratorMorphism (for IsLeftOrRightPresentation, IsInt)

▷ `StandardGeneratorMorphism(A, i)` (operation)

**Returns:** a morphism in $\mathrm{Hom}(F,A)$

The argument is an object $A$ in the category of left or right presentations over a homalg ring $R$ with underlying matrix $M$ and an integer $i$. The output is a morphism $F \to A$ given by the $i$-th row or column of $M$, where $F$ is a free left or right presentation of rank 1.

### 1.5.2 CoverByFreeModule (for IsLeftOrRightPresentation)

▷ `CoverByFreeModule(A)` (attribute)

**Returns:** a morphism in $\mathrm{Hom}(F,A)$

The argument is an object $A$ in the category of left or right presentations. The output is a morphism from a free module $F$ to $A$, which maps the standard generators of the free module to the generators of $A$.

## 1.6 Natural Transformations

### 1.6.1 NaturalIsomorphismFromIdentityToStandardModuleLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToStandardModuleLeft(R)` (attribute)

**Returns:** a natural transformation $\mathrm{Id} \to \mathrm{StandardModuleLeft}$

The argument is a homalg ring $R$. The output is the natural isomorphism from the identity functor to the left standard module functor.

### 1.6.2 NaturalIsomorphismFromIdentityToStandardModuleRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToStandardModuleRight(R)` (attribute)

**Returns:** a natural transformation Id → StandardModuleRight

The argument is a homalg ring $R$. The output is the natural isomorphism from the identity functor to the right standard module functor.

### 1.6.3 NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsLeft(R)` (attribute)

**Returns:** a natural transformation Id → GetRidOfZeroGeneratorsLeft

The argument is a homalg ring $R$. The output is the natural isomorphism from the identity functor to the functor that gets rid of zero generators of left modules.

### 1.6.4 NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsRight(R)` (attribute)

**Returns:** a natural transformation Id → GetRidOfZeroGeneratorsRight

The argument is a homalg ring $R$. The output is the natural isomorphism from the identity functor to the functor that gets rid of zero generators of right modules.

### 1.6.5 NaturalIsomorphismFromIdentityToLessGeneratorsLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToLessGeneratorsLeft(R)` (attribute)

**Returns:** a natural transformation Id → LessGeneratorsLeft

The argument is a homalg ring $R$. The output is the natural morphism from the identity functor to the left less generators functor.

### 1.6.6 NaturalIsomorphismFromIdentityToLessGeneratorsRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToLessGeneratorsRight(R)` (attribute)

**Returns:** a natural transformation Id → LessGeneratorsRight

The argument is a homalg ring $R$. The output is the natural morphism from the identity functor to the right less generator functor.

### 1.6.7 NaturalTransformationFromIdentityToDoubleDualLeft (for IsHomalgRing)

▷ `NaturalTransformationFromIdentityToDoubleDualLeft(R)` (attribute)

**Returns:** a natural transformation Id → FunctorDoubleDualLeft

The argument is a homalg ring $R$. The output is the natural morphism from the identity functor to the double dual functor in left Presentations category.

### 1.6.8 NaturalTransformationFromIdentityToDoubleDualRight (for IsHomalgRing)

▷ `NaturalTransformationFromIdentityToDoubleDualRight(R)` (attribute)

**Returns:** a natural transformation Id $\rightarrow$ FunctorDoubleDualRight

The argument is a homalg ring $R$. The output is the natural morphism from the identity functor to the double dual functor in right Presentations category.

# Chapter 2

# Examples and Tests

## 2.1 Annihilator

Example

```
gap> ZZ := HomalgRingOfIntegersInSingular();;
gap> M1 := AsLeftPresentation( HomalgMatrix( [ [ "2" ] ], ZZ ) );;
gap> M2 := AsLeftPresentation( HomalgMatrix( [ [ "3" ] ], ZZ ) );;
gap> M3 := AsLeftPresentation( HomalgMatrix( [ [ "4" ] ], ZZ ) );;
gap> M := DirectSum( M1, M2, M3 );;
gap> Display( Annihilator( M ) );
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A monomorphism in Category of left presentations of Z
gap> M1 := AsRightPresentation( HomalgMatrix( [ [ "2" ] ], ZZ ) );;
gap> M2 := AsRightPresentation( HomalgMatrix( [ [ "3" ] ], ZZ ) );;
gap> M3 := AsRightPresentation( HomalgMatrix( [ [ "4" ] ], ZZ ) );;
gap> M := DirectSum( M1, M2, M3 );;
gap> Display( Annihilator( M ) );
12

A monomorphism in Category of right presentations of Z
```

## 2.2 Intersection of Submodules

Example

```
gap> Q := HomalgFieldOfRationalsInSingular();;
gap> R := Q * "x,y";
Q[x,y]
gap> F := AsLeftPresentation( HomalgMatrix( [ [ 0 ] ], R ) );
<An object in Category of left presentations of Q[x,y]>
gap> I1 := AsLeftPresentation( HomalgMatrix( [ [ "x" ] ], R ) );;
gap> I2 := AsLeftPresentation( HomalgMatrix( [ [ "y" ] ], R ) );;
gap> Display( I1 );
x

An object in Category of left presentations of Q[x,y]
gap> Display( I2 );
y

```

```
An object in Category of left presentations of Q[x,y]
gap> eps1 := PresentationMorphism( F, HomalgMatrix( [ [ 1 ] ], R ), I1 );
<A morphism in Category of left presentations of Q[x,y]>
gap> eps2 := PresentationMorphism( F, HomalgMatrix( [ [ 1 ] ], R ), I2 );
<A morphism in Category of left presentations of Q[x,y]>
gap> kernelemb1 := KernelEmbedding( eps1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> kernelemb2 := KernelEmbedding( eps2 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> P := FiberProduct( kernelemb1, kernelemb2 );;
gap> Display( P );
(an empty 0 x 1 matrix)

An object in Category of left presentations of Q[x,y]
gap> pi1 := ProjectionInFactorOfFiberProduct( [ kernelemb1, kernelemb2 ], 1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> composite := PreCompose( pi1, kernelemb1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> Display( composite );
x*y

A monomorphism in Category of left presentations of Q[x,y]
```

## 2.3 Koszul Complex

Example

```
gap> Q := HomalgFieldOfRationalsInSingular();;
gap> R := Q * "x,y,z";;
gap> M := HomalgMatrix( [ [ "x" ], [ "y" ], [ "z" ] ], 3, 1, R );;
gap> Ml := AsLeftPresentation( M );;
gap> eps := CoverByFreeModule( Ml );;
gap> iota1 := KernelEmbedding( eps );;
gap> Display( iota1 );
x,
y,
z

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota1 ) );
0, -z,y,
-z,0, x,
-y,x, 0

An object in Category of left presentations of Q[x,y,z]
gap> pi1 := CoverByFreeModule( Source( iota1 ) );;
gap> d1 := PreCompose( pi1, iota1 );;
gap> Display( d1 );
x,
y,
z

A morphism in Category of left presentations of Q[x,y,z]
```

```
gap> iota2 := KernelEmbedding( d1 );;
gap> Display( iota2 );
0, -z,y,
-z,0, x,
-y,x, 0

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota2 ) );;
x,-y,z

An object in Category of left presentations of Q[x,y,z]
gap> pi2 := CoverByFreeModule( Source( iota2 ) );;
gap> d2 := PreCompose( pi2, iota2 );;
gap> Display( d2 );
0, -z,y,
-z,0, x,
-y,x, 0

A morphism in Category of left presentations of Q[x,y,z]
gap> iota3 := KernelEmbedding( d2 );;
gap> Display( iota3 );
x,-y,z

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota3 ) );
(an empty 0 x 1 matrix)

An object in Category of left presentations of Q[x,y,z]
gap> pi3 := CoverByFreeModule( Source( iota3 ) );;
gap> d3 := PreCompose( pi3, iota3 );;
gap> Display( d3 );
x,-y,z

A morphism in Category of left presentations of Q[x,y,z]
gap> N := HomalgMatrix( [ [ "x" ] ], 1, 1, R );;
gap> Nl := AsLeftPresentation( N );;
gap> d2Nl := TensorProductOnMorphisms( d2, IdentityMorphism( Nl ) );;
gap> d1Nl := TensorProductOnMorphisms( d1, IdentityMorphism( Nl ) );;
gap> IsZero( PreCompose( d2Nl, d1Nl ) );
true
gap> cycles := KernelEmbedding( d1Nl );;
gap> boundaries := ImageEmbedding( d2Nl );;
gap> boundaries_in_cyles := LiftAlongMonomorphism( cycles, boundaries );;
gap> homology := CokernelObject( boundaries_in_cyles );;
gap> LessGenFunctor := FunctorLessGeneratorsLeft( R );;
gap> homology := ApplyFunctor( LessGenFunctor, homology );;
gap> StdBasisFunctor := FunctorStandardModuleLeft( R );;
gap> homology := ApplyFunctor( StdBasisFunctor, homology );;
gap> Display( homology );
z,
y,
x
```

```
An object in Category of left presentations of Q[x,y,z]
```

## 2.4 Closed Monoidal Structure

Example

```
gap> R := HomalgRingOfIntegers( );;
gap> M := AsLeftPresentation( HomalgMatrix( [ [ 2 ] ], 1, 1, R ) );
<An object in Category of left presentations of Z>
gap> N := AsLeftPresentation( HomalgMatrix( [ [ 3 ] ], 1, 1, R ) );
<An object in Category of left presentations of Z>
gap> T := TensorProductOnObjects( M, N );
<An object in Category of left presentations of Z>
gap> Display( T );
[ [  3 ],
  [  2 ] ]

An object in Category of left presentations of Z
gap> IsZero( T );
true
gap> H := InternalHomOnObjects( DirectSum( M, M ), DirectSum( M, N ) );
<An object in Category of left presentations of Z>
gap> Display( H );
[ [   0,   0,   0,  -2 ],
  [   1,   2,   0,   0 ],
  [   0,   2,   2,   0 ],
  [   2,   3,   0,   2 ] ]

An object in Category of left presentations of Z
gap> alpha := StandardGeneratorMorphism( H, 3 );
<A morphism in Category of left presentations of Z>
gap> l := LambdaElimination( DirectSum( M, M ), DirectSum( M, N ), alpha );
<A morphism in Category of left presentations of Z>
gap> IsZero( l );
false
gap> Display( l );
[ [  -2,   6 ],
  [  -1,  -3 ] ]

A morphism in Category of left presentations of Z
```

## 2.5 Projectivity test

Example

```
gap> Q := HomalgFieldOfRationalsInSingular();;
gap> R := Q * "x";;
gap> F := FreeLeftPresentation( 2, Q );;
gap> HasIsProjective( F ) and IsProjective( F );
true
gap> G := FreeRightPresentation( 2, Q );;
gap> HasIsProjective( G ) and IsProjective( G );
```

```
true
gap> M := AsLeftPresentation( HomalgMatrix( "[ x, x ]", 1, 2, R ) );;
gap> IsProjective( M );
false
gap> N := AsLeftPresentation( HomalgMatrix( "[ 1, x ]", 1, 2, R ) );;
gap> IsProjective( N );
true
```

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